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AUTHOR(S):

SAITO, Kazuyuki

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EMBEDDABLE AW*-ALGEBRAS AND RELATED TOPICS

東北大理 斎藤 和之
(Kazuyuki SAITÔ)

One of the interesting problems in operator algebras is to find the essential condition for a C*-algebra to have a faithful representation as an embeddable AW*-algebra, where "embeddable" means that it is embedded as a double commutant in a type I AW*-algebra, with the same centre.

In [15], the final conclusion of the above problem was given by showing that

AW*-algebras with a separating family of centre-valued states, which are completely additive on projectins, are embeddable.

(See also [6], [18], [14], and [5]).

In this note, we would like to make a survey of the development of the problem of embeddability of AW*-algebras, with an outline of their proofs. We also would like to give an application of our result to the type determination problem of regular completions. (See [13] for details).

Let us recall that a C*-algebra A is an AW*-algebra if

- (1) each maximal abelian *-subalgebra of A is generated by its projections,
- (2) each family of orthogonal projections $\{e_\alpha\}$ in A has a supremum $\sum_A e_\alpha$ in $\text{Proj}(A)$ (the set of all projections in A).

Kaplansky introduced AW*-algebras and obtained their theory.

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In particular, by an elegant algebraic methods, he extended the Murray-von Neumann's type theory, classification of von Neumann algebras to these more general C^* -algebras.

(*) Are all AW^* -algebras embeddable ?

In 1970, Takenouchi and Dyer, independently, showed that there is an AW^* -factor, which is not a von Neumann algebra and Saitô showed that their AW^* -factors are of type III. Maitland Wright also gave an example of a non embeddable AW^* -factor of type III. [19], [16] and [22].

So our next problem is this.

(**) Are all type II AW^* -algebras embeddable ?

Since every embeddable type II_1 AW^* -algebra has a centre-valued trace ([4]) and every semi-finite AW^* -algebra A , with a faithful finite projection e in A such that eAe has a centre-valued trace, is embeddable ([3]), the question (**) is equivalent to the following, well-known, long standing open problem:

(***) Have all AW^* -algebras of type II_1 got centre-valued traces ?

Several authors ([4], [7], [12], [13], [21], [23]) have considered this question. Maitland Wright, above all, proved that every II_1 AW^* -factor with a strictly positive functional is a von Neumann algebra (and so has a trace) ([21]).

Recently, Ozawa established a transfer principle from von Neumann algebras in Boolean-valued set theory to embeddable AW*-algebras. This transfer principle will, then suggest the following:

Theorem 1. Let A be an AW*-algebra with the centre Z. Suppose that A is finite and has a faithful Z-valued state. Then A is embeddable and has a centre-valued trace.

Corollary 1. Let A be a semi-finite (in particular, type II_∞) AW*-algebra such that A has a faithful Z-valued state. Then A is embeddable.

We can apply Corollary 1 to the following solution of the problem of type determination of the regular completions of separable C*-algebras.

Theorem 2. Let B be a separable C*-algebra and let \hat{B} be its regular completion. Then \hat{B} has no type II direct summands.

Theorem 3. If B is NGCR, then \hat{B} never be embeddable (\hat{B} is of type III also).

Definition 1. Let B be a C*-algebra. We say that B has a countable order dense subset if, there is a countable subset $\{ a_n \}$ in the hermitian part B_h of B, such that,

$$x = \text{LUB}_{B_h} \{ a_n \mid a_n \leq x \}$$

for every $x \in B_h$.

The lemma which links our Corollary 1 with Theorem 2 and Theorem 3 is the following:

Lemma 1. Let A be an embeddable AW^* -algebra. Suppose that A has a countable order dense subset. Then A is of type I.

1. Outline of a proof of Theorem 1.

We can do this by using standard methods of set theory. Main idea is to use Widom's theory of embeddable AW^* -algebras and generalized GNS-construction relative to centre-valued states (See [20]).

Let A be a unital C^* -algebra and let Z be the centre of A . A centre-valued state ϕ on A is a positive Z -linear map from A into Z such that $\|\phi(1)\| \leq 1$.

Recall that a Kaplansky-Hilbert module M , over an abelian AW^* -algebra Z , has a Z -valued inner product \langle , \rangle such that

$$\xi \longrightarrow \|\langle \xi, \xi \rangle\|^{1/2}$$

defines a Banach space norm on M . If M is faithful (in the sense that if $z \in Z$ satisfies $z\xi = 0$ for all $\xi \in M$, then $z = 0$), then the set $L_Z(M)$ of all bounded module endomorphisms of M is a type I AW^* -algebra with centre Z . Conversely, given a type I AW^* -algebra B , with the centre Z , one can construct a faithful Kaplansky-Hilbert module N , over Z , such that $B \cong L_Z(N)$. In this case, for every non-zero $a \in L_Z(N)$

there is $\xi \in N$ such that

$$\langle a^*a\xi, \xi \rangle \neq 0.$$

Just as a numerical state on a C^* -algebra, if A is an AW^* -algebra and ϕ is a centre-valued state, then one can construct a Kaplansky-Hilbert module M_ϕ , a unital $*$ -homomorphism π_ϕ from A into $L_Z(M_\phi)$ and a vector ξ_ϕ in M_ϕ such that

$$\phi(x) = \langle \pi_\phi(x)\xi_\phi, \xi_\phi \rangle$$

for all $x \in A$.

Let $S_Z(A)$ be the set of all Z -valued states on A . Let

$$\pi_A(x) = \oplus \{ \pi_\phi(x) \mid \phi \in S_Z(A) \},$$

and

$$M = \oplus \{ M_\phi \mid \phi \in S_Z(A) \}$$

as a Kaplansky-Hilbert module. Then M is a faithful Kaplansky-Hilbert module over Z and π_A is a faithful Z -representation of A in $L_Z(M)$. In fact, if A is an AW^* -algebra, then one can easily check that there are sufficiently many elements in $S_Z(A)$.

Let $A = \pi_A(A)$ in $L_Z(M)$ and let A'' be the double commutant of A in $L_Z(M)$.

Definition 2. A projection p in A'' is called open if there exists an increasing net $\{a_\alpha\}$ of non-negative elements in A such that

$$\langle a_\alpha \xi, \xi \rangle \uparrow \langle p\xi, \xi \rangle \text{ in } Z_h,$$

for every $\xi \in M$.

A projection q in A'' is called closed if $1 - q$ is open.

Lemma 2. Any projection p in A'' has the smallest closed projection in A'' which dominates p .

Definition 3. Any given projection p in A'' , the above smallest closed projection which dominates p is called the closure of p and denote it by \bar{p} .

Definition 4. A projection p in A'' is called nowhere dense if there is a decreasing net L of projections in A such that $\pi_A(q) \geq \bar{p}$ for all q in L and $\Pi_A\{q \mid q \in L\} = 0$ in $\text{Proj}(A)$.

Lemma 3. Let A be a finite AW*-algebra with the centre Z . Let $\{p_n\}$ be a sequence of nowhere dense closed projections in A'' . Then, $\overline{\sum A'' p_n}$ (in A'') is also nowhere dense.

Let, for each n , L_n be a decreasing net of projections in A such that $\pi_A(q) \geq \bar{p}_n$ for all $q \in L_n$ and $\Pi_A\{q \mid q \in L_n\} = 0$.

The key point of the proof is to construct a decreasing net $\{q_\alpha\}$ of projections in A such that $\pi_A(q_\alpha) \geq \overline{\sum A'' p_n}$ for each α and $\Pi_A\{q_\alpha \mid \alpha\} = 0$ via $\{L_n\}$. See the details for [13].

Next we shall sketch the proof of Theorem 1. Let ϕ be a faithful centre-valued state. Then by our construction of (π_A, M) there is a ξ_ϕ in M such that

$$\phi(x) = \langle \pi_A(x) \xi_\phi, \xi_\phi \rangle$$

for every $x \in A$.

Let ψ be a normal Z -valued state on A'' defined by $\psi(x) = \langle x\xi_\phi, \xi_\phi \rangle$ for every $x \in A''$, then $(\psi|A) \circ \pi_A = \phi$

Let P be the set of all closed nowhere dense projections in A'' and let

$$m = \text{LUB} \{ \psi(p) | p \in P \} \text{ in } Z$$

(Note that $\{ \psi(p) | p \in P \}$ is a bounded subset of Z). Then one can find an increasing net $\{ p_\alpha \}$ in P such that $\psi(p_\alpha) \uparrow m$ (in Z). In fact, if p_1 and $p_2 \in P$, then, by Lemma 3, $\overline{p_1 \vee p_2} \in M$, and so P is an increasing net. Take any non-zero projection z in Z and any positive number ϵ , one can choose an element $p(z) \in P$ and a non-zero subprojection z_1 in Z of z , such that

$$\| (m - \psi(p(z)))z_1 \| < \epsilon.$$

So, we can choose an orthogonal family of central projections $\{ z_\lambda \}$ such that $\sum_\lambda z_\lambda = 1$ and a family $\{ p_\lambda \}$ in P such that

$$\| (m - \psi(p_\lambda))z_\lambda \| < \epsilon$$

for all λ . Let $p = \sum_A p_\lambda z_\lambda$, then

$$\| m - \psi(p) \| < \epsilon.$$

Next, we shall show that $p \in P$. Let L_λ be a decreasing net of projections in A such that $\pi_A(q) \geq \overline{p_\lambda}$ for all $q \in L_\lambda$ and $\pi_A\{q | q \in L_\lambda\} = 0$ for each λ . Let Z be the set $\{ \sum_A z_\lambda q_\lambda | q_\lambda \in L_\lambda \text{ for each } \lambda \}$. Then Z is a decreasing net of projections in A such that $\pi_A(\sum_A z_\lambda q_\lambda) \geq \overline{p}$. Since we can easily check that

$$\pi_A\{ \sum_A q_\lambda z_\lambda | (q_\lambda) = q \in \pi_\lambda L_\lambda \} = 0,$$

Z meets all the requirements for the nowhere density of p .

Because p is closed, $p \in P$ follows.

Thus one can find a sequence $\{p_n\}$ in P such that

$$\|\psi(p_n) - m\| \rightarrow 0 \quad (n \rightarrow \infty).$$

Let $d = \overline{\sum_{A''} p_n}$ ($\in A''$), then, by Lemma 3, $d \in P$ and so $m \geq \psi(d) \geq \psi(p_n)$ ($n = 1, 2, \dots$), which implies that $m = \psi(d)$. Put

$$\eta : A'' \rightarrow Z$$

by

$$\eta(x) = \psi((1-d)x(1-d)) \quad x \in A'',$$

then η is a normal Z -valued non-zero state on A'' . Since $\eta(1) = \psi(1-d) = \psi(1) - \psi(d) = \phi(1) - m$, if $\eta(1) = 0$, then $m = \phi(1)$. On the other hand, $1-d$ is open, there is an increasing net $\{a_\alpha\}$ in A^+ such that

$$\langle \pi_A(a_\alpha)\xi, \xi \rangle \uparrow \langle (1-d)\xi, \xi \rangle \text{ for all } \xi \in M$$

and so

$$\phi(a_\alpha) = \langle \pi_A(a_\alpha)\xi_\phi, \xi_\phi \rangle \uparrow \langle (1-d)\xi_\phi, \xi_\phi \rangle,$$

thus, the faithfulness of ϕ tells us that $a_\alpha = 0$ for all α . Thus it follows that $d = 1$. This is a contradiction. Hence $\eta(1) \neq 0$.

Let $p \in P$ and let $r = \overline{p \vee d}$, then $r \in P$ by Lemma 3. Since $(1-d)p(1-d) \leq r - d$, $\eta(p) \leq \psi(r-d) = \psi(r) - \psi(d) = m - m = 0$.

Let $\{q_\lambda\}$ be any downward directed subset in $\text{Proj}(A)$ with $\Pi_A\{q_\lambda \mid \lambda\} = 0$. Then, $\Pi_{A''}\{\pi_A(q_\lambda) \mid \lambda\} \in P$, which implies that

$$\text{GLB}_\lambda \eta(\pi_A(q_\lambda)) = \eta(\Pi_{A''}\{\pi_A(q_\lambda) \mid \lambda\}) = 0$$

because η is normal on A'' .

Hence, if we put $\phi_0 = \eta|_{\pi_A(A)} \circ \pi_A$, then ϕ_0 is a non-zero Z -valued state of A and normal on $\text{Proj}(A)$. Considering the complement of the support of ϕ_0 if necessary, one can construe sufficiently many Z -valued states $\{\phi_\alpha\}$, each of which is completely additive on $\text{Proj}(A)$. Thus, by a theorem of Widom [20], A is embeddable and so A has a centre-valued trace by [4].

In [3], Elliott, Saitô and Wright showed the following:

Let A be a semi-finite AW*-algebra with the centre Z and let e be a finite projection in A whose central cover is 1. Then A is embeddable if, and only if, the finite corner eAe has a centre-valued trace.

So our Corollary 1 is a direct consequence of the above result.

2. Outline of proofs of Theorem 2 and Theorem 3.

Let \hat{B} be the regular completion of a given separable C^* -algebra B . Then \hat{B} has a countable order dense subset and so \hat{B} has a faithful centre-valued state. If \hat{B} has semi-finite direct summand $\hat{B}e$, then $\hat{B}e$ has a faithful centre-valued state and so $\hat{B}e$ is embeddable. Thus by Lemma 1, $\hat{B}e$ is of type I and hence Theorem 2 follows.

If B is NGCR, then \hat{B} has no type I direct summand ([17]). So the claim in Theorem 3 follows from Theorem 2.

An example. Let A be $F \otimes_{\alpha_0} C[0,1]$, the tensor product of

the Fermion algebra by the C^* -algebra of all complex-valued continuous functions on $[0,1]$; then one can easily show that A is a separable NGCR algebra such that \hat{A} is not embeddable. Moreover, the centre Z of \hat{A} is $*$ -isomorphic to $D[0,1]$, the abelian AW^* -algebra obtained from the regular completion of $C[0,1]$.

3. Appendix. Towards the additivity of traces in finite AW^* -algebras.

In this section, we shall give a sufficient condition for the additivity of traces, which, we hope, will play a role in solving this problem.

Let D be the unique dimension function on a type II_1 AW^* -factor A . Let $a \in A_h$ with $a = \int \lambda de_\lambda$. Put

$$\text{Tr}(a) = \int \lambda dD(e_\lambda),$$

then Tr satisfies the following properties:

- (1) $\text{Tr}(u*au) = \text{Tr}(a)$ for all $a \in A_h$, $u \in A_u$,
- (2) $\text{Tr}(a) \geq 0$ for all $a \in A^+$
- (3) $\text{Tr}(\lambda a) = \lambda \text{Tr}(a)$ for all $\lambda \in \mathbb{R}$ and $a \in A_h$,
- (4) $\text{Tr}(x+y) = \text{Tr}(x) + \text{Tr}(y)$ if $x, y \in A_h$ and $xy = yx$.

(****) Can we conclude that

$$\text{Tr}(x + y) = \text{Tr}(x) + \text{Tr}(y)$$

for any pair x and y in A_h ?

Proposition 1. Let A be a type II_1 AW^* -factor and let Tr be the above defined restricted trace on A . If Tr satisfies that

$$(5) \quad \text{Tr}(e_1 + e_2 + e_3) = \text{Tr}(e_1) + \text{Tr}(e_2) + \text{Tr}(e_3)$$

for every triplet $\{e_1, e_2, e_3\}$ in $\text{Proj}(A)$, then the answer to (****) is positive.

Lemma 4 (T. Ono). If Tr satisfies (5), then, for any projection e and any pair $\{e_1, e_2\}$ of orthogonal projections in A . $\text{Tr}(e(e_1 + e_2)e) = \text{Tr}(ee_1e) + \text{Tr}(ee_2e)$.

Let $e_3 = 1 - e_1 - e_2$ and let ω be the primitive root of $x^3 = 1$. Take $v = e_1 + \omega e_2 + \omega^2 e_3$, then v is a unitary in A such that $e + v*ev + v^2*ev^2 = 3(e_1ee_1 + e_2ee_2 + e_3ee_3)$ and so

$$\begin{aligned} \text{Tr}(e + v*ev + v^2*ev^2) &= \text{Tr}(e) + \text{Tr}(v*ev) + \text{Tr}(v^2*ev^2) \\ &= 3\text{Tr}(e). \end{aligned}$$

Hence it follows that $\text{Tr}(e) = \text{Tr}(e_1ee_1) + \text{Tr}(e_2ee_2) + \text{Tr}(e_3ee_3)$.

On the other hand, the fact that $\text{Tr}(e_1ee_1) + \text{Tr}(ee_1e)$ for each i , tells us that $\text{Tr}(e) = \text{Tr}(ee_1e) + \text{Tr}(ee_2e) + \text{Tr}(ee_3e)$.

Observe that $\text{Tr}|_{eAe}$ is a restricted trace, it follows that

$$\begin{aligned} \text{Tr}(ee_3e) &= \text{Tr}(e(1 - e_1 - e_2)e) \\ &= \text{Tr}(e - e(e_1 + e_2)e) \\ &= \text{Tr}(e) - \text{Tr}(e(e_1 + e_2)e). \end{aligned}$$

Thus it follows that $\text{Tr}(e(e_1 + e_2)e) = \text{Tr}(ee_1e) + \text{Tr}(ee_2e)$.

Since A is of type II, there are four orthogonal equivalent projections f_1, f_2, f_3 and f_4 such that $f_1 + f_2 + f_3 + f_4 = 1$. By using this system, one can canonically construct 4 by 4 system of matrix units $\{e_{ij}\}$ in A such that $e_{11} = f_1$.

We shall show, under the condition (5), that f_1Af_1 is a von Neumann algebra. Let a and b be in $(f_1Af_1)^+$ such that

$a \leq (1/2)1$ and $b \leq (1/2)1$. Let

$$p = \begin{pmatrix} a & a & (a - 2a^2)^{1/2} & 0 \\ a & a & (a - 2a^2)^{1/2} & 0 \\ (a - 2a^2)^{1/2} & (a - 2a^2)^{1/2} & 1 - 2a & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$q = \begin{pmatrix} b & -b & 0 & -(b - 2b^2)^{1/2} \\ -b & b & 0 & (b - 2b^2)^{1/2} \\ 0 & 0 & 0 & 0 \\ -(b - 2b^2)^{1/2} & (b - 2b^2)^{1/2} & 0 & 1 - 2b \end{pmatrix}$$

via $\{e_{ij}\}$, then p and q are orthogonal projections in A such that $f_1 p f_1 = a$ and $f_1 q f_1 = b$. So, if, Tr satisfies (5), then

$$\text{Tr}(f_1(p + q)f_1) = \text{Tr}(f_1 p f_1) + \text{Tr}(f_1 q f_1),$$

by Lemma 4, which implies that $\text{Tr}(a + b) = \text{Tr}(a) + \text{Tr}(b)$, and so $\text{Tr } f_1 A f_1$ is a trace on $f_1 A f_1$. Hence $f_1 A f_1$ is a von Neumann factor. Thus by the previous arguments before the proof of Corollary 1, we can get that A is a von Neumann factor, that is, Tr is additive. This completes the proof.

References

- [1] G. A. Akemann, The generalized Stone-Weierstrass problem, *J. Funct. Anal.* 4(1969), 277-294.
- [2] E. Christensen, Measures on projections and physical states, *Comm. Math. Phys.* 86(1982), 529-538.
- [3] G. A. Elliott, K. Saitô and J. D. M. Wright, Embedding AW^* -algebras as double commutants in type I algebras, *J. London Math. Soc.*, 28(1983), 376-384.
- [4] M. Goldman, Structure of AW^* -algebras, *Duke Math. J.* 23(1956), 23-34.
- [5] H. Halpern, Embedding as double commutator in a type I AW^* -algebra, *Trans. Amer. Math. Soc.*, 148(1970), 85-98.
- [6] R. V. Kadison, Operator algebras with a faithful weakly closed representation, *Ann. of Math.* 64(1956), 175-181.
- [7] _____, On the additivity of the trace in finite factors, *Proc. Nat. Acad. Sci. U.S.A.*, 41(1955), 385-387.
- [8] I. Kaplansky, Projections in Banach algebras, *Ann. of Math.*, 53(1951), 235-249.
- [9] _____, Algebras of type I, *Ann. of Math.*, 56(1952), 460-472.
- [10] _____, Modules over operator algebras, *Amer. J. Math.*, 75(1953), 839-858.
- [11] M. Ozawa, A transfer principle from von Neumann algebras to AW^* -algebras, *J. London Math. Soc.*, 32(1985), 141-148.
- [12] _____, Boolean valued analysis approach to the trace problem of AW^* -algebras, *J. London Math. Soc.*, 33(1986), 347-354.
- [13] _____ and K. Saitô, Embeddable AW^* -algebras and regular

- completions, J. London Math. Soc., 34(1986), 511-523.
- [14] G. K. Pedersen, Operator algebras with weakly closed abelian subalgebras, Bull. London Math. Soc., 4(1972), 171-175.
- [15] K. Saitô, On the embedding as a double commutator in a type I AW*-algebra II, Tôhoku Math. J. 26(1974), 333-340.
- [16] _____, AW*-algebras with monotone convergence property and type III, non W*, AW*-factors, C*-algebras and applications to Physics, (Springer) Lecture Note, No. 650, 1978, 131-134.
- [17] _____, A structure theory in the regular σ -completion of C*-algebras, J. London Math. Soc., 22(1980), 549-558.
- [18] S. Sakai, A characterization of W*-algebras, Pacific J. Math., 6(1956), 763-773.
- [19] G. Takenouchi, A non W*, AW*-factor, C*-algebras and applications to Physics. (Springer) Lecture Note. No. 650, 1978, 135-139.
- [20] H. Widom, Embedding in algebras of type I, Duke Math. J., 23(1956), 309-324.
- [21] J. D. M. Wright, On AW*-algebras of finite type, J. London Math. Soc., 12(1976), 431-439.
- [22] _____, Wild AW*-factors and Kaplansky-Rickart algebras, J. London Math. Soc., 13(1976), 83-89.
- [23] Ti Yen, Trace on finite AW*-algebras, Duke Math. J. 22(1955), 207-222.

Mathematical Institute

Tôhoku University, Sendai